Dynamic instability in crack propagation

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A linear stability analysis of the steady state solutions to a model for propagating cracks implies dynamic instability beyond a critical crack speed, oscillatory behavior of the crack tip velocity, and the emergence of spatial structure. The results are in qualitative agreement with experiment.

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A well known phenomenon of brittle fracture, studied using longitudinally stressed samples of Plexiglass that have been seeded with a small flaw, involves riblike patterns found on the crack surfaces. The roughly periodic structure emerges from a smooth region, becoming larger in the direction of propagation. A second problem of fracture mechanics concerns the fact that propagating cracks do not attain the limiting Raleigh surface wave speed predicted by linear elasticity theory [1,2].

In their investigation of mode I fracture in Plexiglass, Fineberg, Gross, Marder, and Swinney [2] found that the two phenomena are closely linked. They report that cracks accelerate smoothly up to a critical speed, beyond which the crack tip velocity oscillates about an average value. Structure on the crack surface develops only in conjunction with these oscillations. It was proposed [2] that both phenomena, the patterns and smaller limiting speeds, are manifestations of a dynamical instability.

This hypothesis is borne out by the work presented here: (i) an equation of motion for the crack profile; (ii) steady state solutions of the equation of motion; (iii) a linear stability analysis of these solutions. The model equation and its steady state solutions were originally introduced by Langer [3].

The results of this stability analysis display the properties of fracture mentioned above and provide an understanding of other observations as well. The results are also consistent with the behavior of other continuum systems. For example, the dynamic instability is related to a viscous energy dissipation mechanism, and we find that it occurs only in a regime of sufficiently large effective Reynolds number. This is analogous to situations in fluid mechanics.

In this paper, emphasis is placed on the method and results of the stability analysis. A derivation of the model, which establishes its connection to elasticity theory, will be presented elsewhere.

A detailed presentation of the model was given by Langer [3]. It describes a crack with a cohesive zone [1] at the tip. The crack is driven by elastic energy of the strained material, and energy dissipation takes place by way of a viscous term. In the following summary of the model and its solutions, Eqs. (1)–(9) were taken from Ref. [3].

The differential equation for the displacement $U(x,t)$, which describes the crack profile at position x and time t, is

$$\partial_t^2 U = c^2 \partial_x^2 U - m^2(U-\Delta) - f(U) + \eta \partial_x^2 \partial_t U \ . \quad (1)$$

The cohesive force has the form of a step function,

$$F(U) = \begin{cases} f_0, & 0 \le U \le \delta \\ 0, & \delta < U \ , \end{cases} \quad (2)$$

where δ is the range of cohesion.

With times and lengths scaled so that $c=1$, the steady state solutions $U(\xi)$, where $\xi = x + vt$, are found by solving

$$\eta v U''' + (1-v^2)U'' - m^2(U-\Delta) = f(U) \ . \quad (3)$$

Figure 1 shows the quantities characterizing a crack [a solution to (3)] moving with velocity $-v$.

In front of the tip the displacement is identically zero; far behind the tip the displacement approaches Δ, a measure of the applied force or strain. The length of the cohesive zone (I) is l.

Equation (3) is piecewise linear and its solutions in I and II have the form

$$U = \text{const} + \sum_j A_j \exp(q_j \xi) \ , \quad (4)$$

where the q_j are roots of

$$\eta v q^3 + (1-v^2)q^2 - m^2 = 0 \ . \quad (5)$$

The term containing the positive root of (5), which is defined as q_1, is missing in zone II.

For given values of m, η, f_0, δ, and Δ there are the five coefficients A_j, the length of the cohesive zone l and the velocity of the crack v to be determined. The required seven equations are

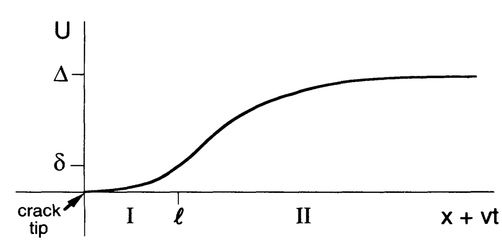

FIG. 1. Schematic picture of the displacement $U(x+vt)$ and the quantities that characterize a crack propagating with velocity $-v$.

$$U^{(n)}(0)=0,\quad U^{(n)}(l+\epsilon)=U^{(n)}(l-\epsilon),\quad U(l)=\delta\ , \tag{6}$$

for $n=0,1,2$. Here $U^{(n)}$ denotes the nth derivative of U.

A particular goal of the calculation in [3] was to determine the relation between crack velocity and applied force—that is, the function $v(\Delta)$—which provides information on the existence and character of the family of steady state modes. To find $v(\Delta)$ one has the equation

$$\exp(-q_1 l)=1-\frac{m^2\Delta}{f_0} \tag{7}$$

and some approximations. For small values of $v\eta$,

$$v\approx\frac{m}{f_0\eta}(\Delta-\Delta_G)\ , \tag{8}$$

with $m^2\Delta_G^2=2f_0\delta$ the Griffith criterion for this system [1,3]. For larger velocities,

$$\frac{\Delta}{\Delta_G}\approx\left[\frac{3\Delta_G}{2\delta}\right]^{1/3}\left[\frac{\eta v}{m^2}\right]^{1/3}q_1(v)\ . \tag{9}$$

Two features of these solutions to the model were stressed in [3]. First, that propagation at speeds in excess of the sound speed seems to be possible at arbitrary values of m and η. Also, it was noted that two of the roots of (5), q_2 and q_3, are complex at speeds greater than the solution v_* of the equation $(1-v^2)^3/v^2=27m^2\eta^2/4$. When the wave numbers $q_{2,3}$ have imaginary components, the crack profile (4) acquires oscillatory structure.

In fact it can be shown from (5), (7), and (9) that $v=1$ is a limiting speed in the following sense: $\max v(\Delta)=1+\epsilon(\eta)$, where ϵ can be made arbitrarily small for small η. The form that $v(\Delta)$ takes in this regime is shown in Fig. 2. Furthermore, minute values of a dissipation coefficient are a natural assumption for brittle fracture, which has often been modeled with no dissipation at all [1]. Since the speed v_* defined above also approaches unity as η decreases, we conclude that supersonic propagation and the oscillatory structure associated with complex wave numbers q_j are features of an unphysical region of parameter space.

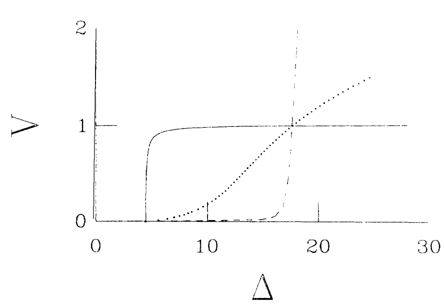

FIG. 2. Three examples of steady state crack velocity v as a function of Δ. The maximum displacement Δ is proportional to the applied force or strain. Times and lengths have been scaled so that the wave speed equals 1. The parameter values that were used in calculating this function are $m^2=1$, $f_0=100$, $\delta=0.1$, and $\eta=100$ (dashed line), $\eta=1$ (dotted line), and $\eta=0.001$ (solid line).

In the preceding paragraphs we discussed the viscous model of crack propagation and its steady state solutions, henceforth referred to as $U_0(x+vt)$. For these exact solutions to occur in nature, they must be stable with respect to infinitesimal perturbations. The investigation of this question will proceed as follows. Substituting $U=U_0+U_1$ into the differential equation (1) and linearizing leads to

$$\partial_t^2 U_1-\partial_x^2 U_1+m^2U_1-\eta\partial_x^2\partial_t U_1=\frac{f_0}{U_0'(l)}\delta(x+vt-1)U_1. \tag{10}$$

Assume that $U_1=U_1(x+vt,t)$ and has continuous first partial derivatives. By integrating (10) with respect to time, over an interval defined by $x+vt=l$, one obtains the boundary condition,

$$\partial_x^2 U_1(l+\epsilon,t)-\partial_x^2 U_1(l-\epsilon,t)=\frac{-f_0}{\eta v U_0'(l)}U_1(l,t)\ . \tag{11}$$

Now, the steady state modes U_0 must be investigated for stability with respect to perturbations of the crack profile *and* the tip speed. Thus we seek solutions $U=U(x+vt+h(t),t)$ in the vicinity of U_0. The derivative of $h(t)$ is equal to the velocity of the crack tip in the moving frame. Expand U to obtain the form of the small quantity,

$$U_1=G(x+vt,t)+U_0'(x+vt)h(t)\ . \tag{12}$$

Note that G describes the perturbation to the shape of the crack profile, while $U_0'h$ corresponds to a small displacement of the entire crack. Using the original differential equation (3), it is easily shown that with a sufficiently smooth $\phi(x,t)$, any function $U_1=U_0'(x+vt)\phi(x,t)$ will satisfy the boundary condition (11). We therefore write

$$U_1=U_0'(x+vt)g(x+vt)e^{\omega t}+hU_0'(x+vt)e^{\omega t}\ . \tag{13}$$

Because U_0' has discontinuous higher (than second) derivatives at $x+vt=0$ and $x+vt=l$, the function g must satisfy $g'/(g+h)=-\omega/3v$ at these two points [see Eq. (14), ff.]. To explicitly evaluate this (necessary) condition, and solve for ω, let $g=b_k e^{ikx+ikvt}$. With $T\equiv\partial_t^2-\partial_x^2+m^2-\eta\partial_x^2\partial_t$, the three equations

$$TU_1(0,t)=0\ , \tag{14}$$

$$TU_1(l+\epsilon,t)=0\ , \tag{15}$$

$$TU_1(l-\epsilon,t)=0\ , \tag{16}$$

allow one to solve for k, h (or b), and ω. From (14) we find that $(3ikv+\omega)b=-\omega h$. Subtracting (15) from (16) results in $(f_0\omega/v)(e^{ikl}-1)=0$. Thus the values of k are restricted to $k=2\pi n/l$. For simplicity, we define $l_*^{-1}\equiv U_0''(l)/U_0'(l)$, where $l_*\sim l$. The relation $\omega=\mathrm{Re}(\omega)+i\,\mathrm{Im}(\omega)$ follows from (15) or (16).

The wave speed c has been inserted into the following expressions, which give the central result of the calculation, $\omega=N^{-1}(A+iB)$, with

$$N=\eta^2k^2+\left[v+\frac{2\eta}{l_*}\right]^2 , \quad (17)$$

$$A=-\eta k^2\left[c^2-2v^2+\frac{\eta v}{l_*}\right]-2(2v^2+c^2)\frac{1}{l_*}\left[v+\frac{2\eta}{l_*}\right] , \quad (18)$$

$$B=-vk\left[\eta^2k^2+c^2-v^2-\frac{3\eta v}{l_*}+\frac{6\eta^2}{l_*^2}\right] . \quad (19)$$

As stated here, the exact expression for ω appears to be rather complicated. Fortunately, it simplifies in the regime of small η.

We now turn to the analysis of ω, noting again that it derives from an attempt to evaluate a necessary condition, not a full solution of the differential equation. As is evident from (13), negative values of its real part describe damping of the perturbation, while positive values imply instability of the steady state mode. The imaginary part of ω is the frequency with which the crack speed oscillates about its steady state value. For sufficiently small values of the dissipation constant η we will find instability, oscillation, and spatial structure.

At this point it is useful to state some approximations that are valid in the regime of small η. As shown in Fig. 2, $v(\Delta)$ becomes very steep as η decreases. For crack speeds less than c, $\Delta\approx\Delta_G$. The positive root of (5) will be $q_1\approx m/\sqrt{c^2-v^2}$ and if $m^2\Delta_G/f_0\ll 1$ we have from (7) and the definition of Δ_G,

$$l\approx\frac{m^2\Delta_G}{q_1 f_0}\approx\sqrt{2\delta/f_0}\sqrt{c^2-v^2} . \quad (20)$$

Thus the number $\eta v/l_* c^2$ can be made arbitrarily small. Note that this corresponds to the regime in which the effective Reynolds number $l_* c/\eta$ takes on large values.

Inspection of (18) reveals the necessary condition for instability:

$$1-\frac{2v^2}{c^2}+\frac{\eta v}{l_* c^2}<0 . \quad (21)$$

The positive root of this expression defines a critical velocity v_c. In the regime under investigation, $v_c\approx c/\sqrt{2}$ and the dispersion relation takes on a more inviting form. Near v_c,

$$\omega\approx\frac{4v_c}{\eta^2k^2+v_c^2}[\eta k^2(v-v_c)-(c^2/l_*)]-ivk . \quad (22)$$

Recall that the perturbation of the shape of the crack profile was represented by a Fourier mode $b_k e^{ik\xi}$. With (22), $ge^{\omega t}\propto e^{ik\xi-ikvt}e^{\mathrm{Re}(\omega t)}$ describes a perturbation moving to the right with velocity v, a localized structure in the laboratory frame.

Suppose now that v is only slightly greater than v_c. Then instability sets in when the perturbation has a very large wave number k. In this sense, there is a critical wavelength that increases with the velocity of the crack:

$$\lambda(v)\propto\eta^{1/2}(v-v_c)^{1/2} . \quad (23)$$

An increase in the wavelength of structure with crack speed has been observed in fracture experiments [2].

The imaginary part of ω describes the oscillatory frequency of the tip speed in the moving frame; $\Omega\equiv-\mathrm{Im}(\omega)\approx vk$. It was pointed out by Fineberg *et al.* that elastic theory does not naturally provide the time scale corresponding to the observed oscillations [2]. This question can be addressed in the context of the present work. The values reported by Fineberg *et al.* for period of oscillations, crack velocity, and wavelength of structure are in accordance with the relation $\Omega\sim vk=v2\pi n/l$. Furthermore, near the critical speed, the length of the cohesive zone is $l\approx c\sqrt{\delta/f_0}$, which makes $\tau\equiv\sqrt{\delta/f_0}$ a natural time scale of the system and its oscillations.

Consider the physical situation depicted in Fig. 3. An approximation of f_0 as the derivative of stress suggests that

$$\tau\equiv\left[\frac{\delta}{f_0}\right]^{1/2}\sim\frac{\delta^{1/2}(\text{density})^{1/2}}{(\text{tensile strength})^{1/2}}H^{1/2} , \quad (24)$$

where H is the height of the sample. This expression yields a simple dependence of the time scale on sample geometry.

The results of this analysis allow predictions of fracture phenomena that can be experimentally investigated. First, it is necessary to stress that we have been discussing cracks propagating with constant velocity, whereas brittle fracture typically involves large accelerations. It is therefore desirable to obtain data from experiments in which accelerations are systematically reduced. Under these conditions, one can expect to find critical velocities that approach but do not exceed $c_R/\sqrt{2}$, approximately seven-tenths of the Raleigh wave speed.

Apparently, $(v-v_c)/v_c$ can be taken as a control parameter of the instability; v is a monotonic function of Δ, and Δ represents the applied force or strain. Following Landau [4], one expects the amplitude of the oscillatory structure that emerges, also perpendicular to the direction of propagation, to increase as $A_{\max}\propto[(v-v_c)/v_c]^{1/2}$. It was reported in [2] that the measured amplitude is linear in this parameter. However, the published data also carry a hint of the square root behavior; again, the prediction is for steady state states, and "smoother" experiments may clarify the situation.

According to Eq. (23), materials with lower viscous energy dissipation will develop smaller structure in the unstable region, and because $\Omega\approx vk$, the frequency of the

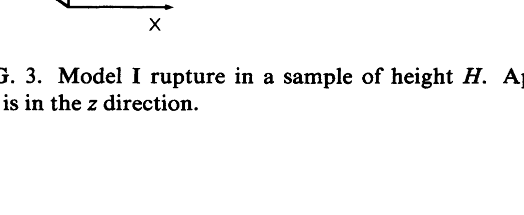

FIG. 3. Model I rupture in a sample of height H. Applied stress is in the z direction.

corresponding oscillations will be higher. Both of these tendencies have been observed in fracture experiments. In Plexiglass samples of varying molecular weight, Kusy and Turner found a marked decrease in the size of oscillatory structure as the molecular weight of the polymer was reduced [6]. The reduction of molecular weight however, which causes the samples to become more brittle and glasslike [6], clearly implies less viscous energy dissipation. More recently, fracture experiments with glass and Plexiglass samples of the same size showed significantly higher oscillatory frequencies in the glass samples [7].

The nature of the transition to instability can be investigated without having access to crack velocity as a function of position. Square root behavior near the critical velocity, of both the amplitude and the wavelength, implies a linear relation between these latter quantities. A dependence of time and length scales on sample size, as given by (24) and Fig. 3, might also be easy to test.

We have shown that the unstable behavior of this crack propagation model corresponds to some phenomena observed in fracture experiments. The existence and character of the instability appear to be features of a model which incorporates a viscous energy dissipation mechanism. It should be noted that this is a general form of internal friction in continuum systems. When "energy dissipation is not considerable [5]," it follows directly from the dissipative stress tensor of elasticity theory. It is as such a natural choice for a continuum description of brittle fracture.

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